MathAA_11_SL_Summer_2021_Q1

Solution

1. Analysis of the function $f(x)$

Based on the provided graph of $y = f(x)$ for the domain $-4 \leq x \leq 6$:

- **(a) (i) Finding $f(2)$:** Locate $x = 2$ on the horizontal axis. Moving vertically to the curve, the corresponding y-coordinate is 6.

$$f(2) = 6$$

- **(a) (ii) Finding $(f \circ f)(2)$:** The **composite function** is defined as $f(f(2))$. Using the result from the previous step:

$$\begin{aligned}(f \circ f)(2) &= f(f(2)) \\ &= f(6)\end{aligned}$$

  Locate $x = 6$ on the horizontal axis. Moving vertically to the curve, the corresponding y-coordinate is -2.

$$(f \circ f)(2) = -2$$

2. Sketching the graph of $g(x)$

The function $g(x) = \frac{1}{2}f(x) + 1$ is a **transformation** of $f(x)$ involving a vertical compression by a factor of $\frac{1}{2}$ followed by a vertical translation up by 1 unit.

- **Mapping key points $(x, f(x)) \to (x, \frac{1}{2}f(x) + 1)$:**
  - For $-4 \leq x \leq 0$, $f(x) = 4$.

    $$g(x) = \frac{1}{2}(4) + 1 = 3$$
  - At $x = 2$, $f(2) = 6$.

    $$g(2) = \frac{1}{2}(6) + 1 = 4$$
  - At $x = 4$, $f(4) = 4$.

    $$g(4) = \frac{1}{2}(4) + 1 = 3$$
  - At $x \approx 5.5$ (x-intercept), $f(5.5) = 0$.

    $$g(5.5) = \frac{1}{2}(0) + 1 = 1$$
  - At $x = 6$, $f(6) = -2$.

    $$g(6) = \frac{1}{2}(-2) + 1 = 0$$

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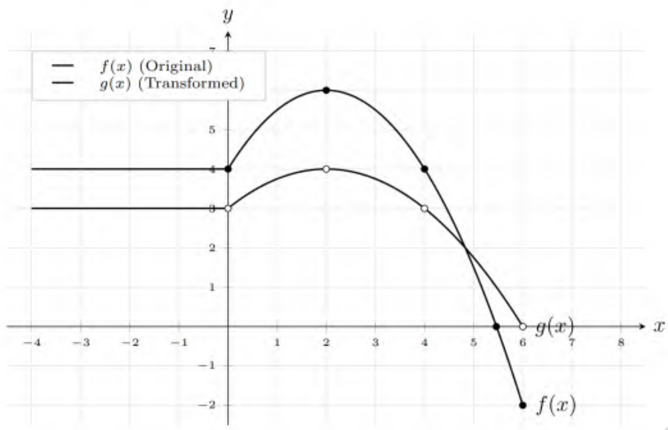

(a) (i) $\boxed{6}$

(b) (ii) $\boxed{-2}$

(c) The graph of $g(x)$ is a horizontal line at $y = 3$ for $x \in [-4, 0]$, and a curve with a maximum at $(2, 4)$ ending at $(6, 0)$.

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MathAA_11_SL_Summer_2021_Q2

Solution

1. Determining the Radius of the Planet

The **radius** r of a sphere is defined as half of its **diameter** d. Given the diameter $d = 6 \times 10^4\,\mathrm{km}$, we calculate the radius as follows:

$$\begin{aligned} r &= \frac{d}{2} \\ &= \frac{6 \times 10^4\,\mathrm{km}}{2} \\ &= 3 \times 10^4\,\mathrm{km} \end{aligned}$$

2. Calculating the Volume and Scientific Notation Parameters

The volume V of a **sphere** is given by the formula:

$$V = \frac{4}{3}\pi r^3$$

- Substitute the radius $r = 3 \times 10^4\,\mathrm{km}$ into the volume formula:

$$\begin{aligned} V &= \frac{4}{3}\pi(3 \times 10^4\,\mathrm{km})^3 \\ &= \frac{4}{3}\pi(3^3 \times (10^4)^3)\,\mathrm{km}^3 \\ &= \frac{4}{3}\pi(27 \times 10^{12})\,\mathrm{km}^3 \\ &= \pi(4 \times 9 \times 10^{12})\,\mathrm{km}^3 \\ &= \pi(36 \times 10^{12})\,\mathrm{km}^3 \end{aligned}$$

- The problem requires the volume to be expressed in the form $\pi(a \times 10^k)\,\mathrm{km}^3$ where $1 \le a < 10$ and $k \in \mathbb{Z}$ (**scientific notation**). We convert 36×10^{12} to this form:

$$\begin{aligned} 36 \times 10^{12} &= (3.6 \times 10^1) \times 10^{12} \\ &= 3.6 \times 10^{13} \end{aligned}$$

- Comparing $\pi(3.6 \times 10^{13})$ to the required form $\pi(a \times 10^k)$, we identify:
  - $a = 3.6$
  - $k = 13$

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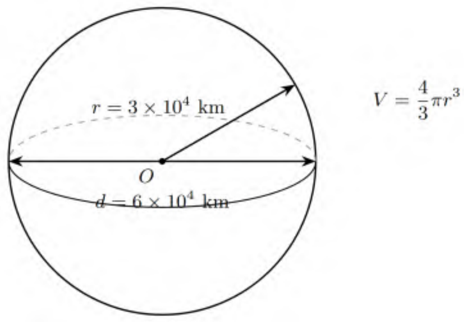

Spherical Planet Model

Final Answer:

(a) The radius of the planet is $\boxed{3 \times 10^4 \, \text{km}}$.

(b) The values are $\boxed{a = 3.6, k = 13}$.

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MathAA_11_SL_Summer_2021_Q3

Solution

To find the first term u_1 and the **common difference** d of the **arithmetic sequence**, we utilize the standard formulas for the n-th term and the sum of the first n terms.

1. Establishing the equations The n-th term of an arithmetic sequence is given by:

$$u_n = u_1 + (n-1)d$$

The sum of the first n terms is given by:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

Given the conditions $u_8 = 8$ and $S_8 = 8$, we can substitute $n = 8$ into these formulas:

- From $u_8 = 8$:

  $u_1 + 7d = 8 \quad ...(1)$

- From $S_8 = 8$:

  $$\frac{8}{2}(u_1 + u_8) = 8$$
  $$4(u_1 + 8) = 8$$

2. Solving for the first term u_1 Using the equation derived from S_8:

$$\begin{aligned} 4(u_1 + 8) &= 8 \\ u_1 + 8 &= \frac{8}{4} \\ u_1 + 8 &= 2 \\ u_1 &= 2 - 8 \\ u_1 &= -6 \end{aligned}$$

3. Solving for the common difference d Substitute the value of $u_1 = -6$ into equation (1):

$$\begin{aligned} -6 + 7d &= 8 \\ 7d &= 8 + 6 \\ 7d &= 14 \\ d &= \frac{14}{7} \\ d &= 2 \end{aligned}$$

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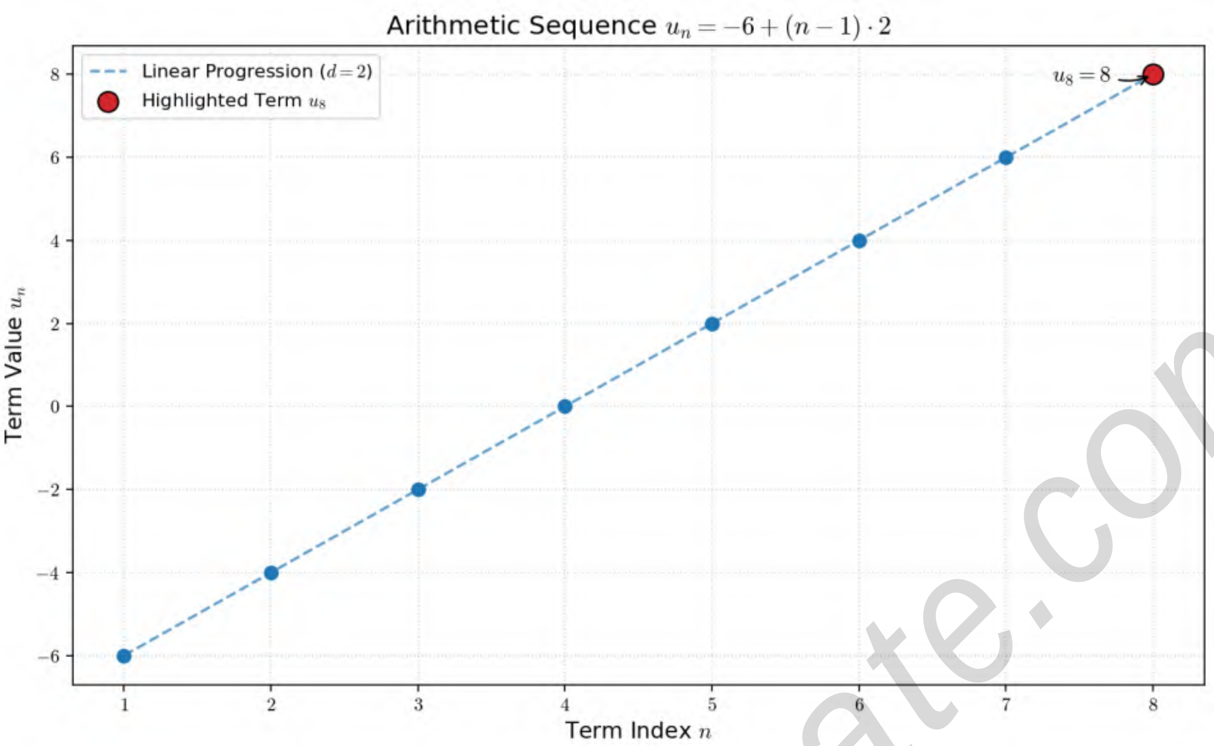

The values for the first term and the common difference are:

$$\boxed{u_1 = -6, d = 2}$$

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MathAA_11_SL_Summer_2021_Q4

Solution

To solve for the values of the **lower quartile** (L) and **upper quartile** (U) from the given **box and whisker diagram**, we utilize the definitions of the five-number summary and the criteria for **outliers**.

1. Identification of known parameters From the diagram and the problem description, we have the following values:

- Minimum value ($x_{\min}$) = $10\,\mathrm{g}$
- Median (Q_2) = $40\,\mathrm{g}$
- Maximum value ($x_{\max}$) = $75\,\mathrm{g}$
- **Interquartile range** (IQR) = $20\,\mathrm{g}$
- No outliers are present in the dataset.

2. Outlier boundaries and the upper quartile An outlier is typically defined as any value that lies outside the range:

$$[L - 1.5 \times IQR, U + 1.5 \times IQR]$$

Since there are no outliers, all data points, including the maximum value, must fall within these boundaries. Specifically, for the upper end:

$$x_{\max} \leq U + 1.5 \times IQR$$

Substituting the known values:

$$\begin{aligned} 75 &\leq U + 1.5(20) \\ 75 &\leq U + 30 \\ U &\geq 75 - 30 \\ U &\geq 45 \end{aligned}$$

Additionally, by the definition of a box plot, the median must be less than or equal to the upper quartile ($Q_2 \leq U$). Since $40 \leq 45$, the constraint from the outlier calculation is the limiting factor. Thus, the minimum possible value for U is 45.

(a) The minimum possible value of U is $\boxed{45}$.

3. Determining the lower quartile The **interquartile range** is defined as the difference between the upper and lower quartiles:

$$IQR = U - L$$

We are given that $IQR = 20$. To find the minimum possible value of L, we rearrange the equation:

$$L = U - IQR$$

To minimize L, we must use the minimum possible value of U found in part (a):

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$$\begin{aligned} L_{\min} &= U_{\min} - 20 \\ &= 45 - 20 \\ &= 25 \end{aligned}$$

We must also check if this L value creates any outliers at the lower end. The lower boundary for outliers is:

$$\begin{aligned} \text{Lower Boundary} &= L - 1.5 \times IQR \\ &= 25 - 1.5(20) \\ &= 25 - 30 \\ &= -5 \end{aligned}$$

Since the minimum value of the data is 10, and $10 > -5$, there are no outliers at the lower end when $L = 25$. Furthermore, $L \leq Q_2$ $(25 \leq 40)$ is satisfied.

(b) The minimum possible value of L is $\boxed{25}$.

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MathAA_11_SL_Summer_2021_Q5

Solution

1. Derivative of $f(x)$

To find the derivative of the function $f(x) = -(x-h)^2 + 2k$, we apply the **power rule** and the **chain rule**. Since h and k are constants:

$$\begin{aligned} f'(x) &= \frac{d}{dx}[-(x-h)^2 + 2k] \\ &= -2(x-h) \cdot \frac{d}{dx}(x-h) + 0 \\ &= -2(x-h) \end{aligned}$$

$$\boxed{f'(x) = -2(x-h)}$$

2. Determining the value of h

The problem states that the graphs of f and g have a **common tangent** at $x = 3$. This implies two conditions at the point of tangency:

- The slopes of the functions must be equal: $f'(3) = g'(3)$.
- The functions must share the same y-coordinate: $f(3) = g(3)$.

First, we find the derivative of $g(x) = e^{x-2} + k$:

$$g'(x) = e^{x-2}$$

Equating the derivatives at $x = 3$:

$$\begin{aligned} f'(3) &= g'(3) \\ -2(3-h) &= e^{3-2} \\ -6 + 2h &= e^1 \\ 2h &= e + 6 \\ h &= \frac{e+6}{2} \end{aligned}$$

This confirms the required expression for h.

3. Determining the value of k

Using the second condition for a common tangent, the function values must be equal at $x = 3$:

$$f(3) = g(3)$$

Substitute the expressions for $f(x)$ and $g(x)$:

$$-(3-h)^2 + 2k = e^{3-2} + k$$

Rearranging to solve for k:

$$k = e + (3-h)^2$$

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From the previous step, we know $h = \frac{e+6}{2}$. We substitute this into the term $(3 - h)$:

$$\begin{aligned} 3 - h &= 3 - \frac{e+6}{2} \\ &= \frac{6 - (e+6)}{2} \\ &= \frac{-e}{2} \end{aligned}$$

Now, substitute this back into the equation for k:

$$\begin{aligned} k &= e + \left(\frac{-e}{2}\right)^2 \\ &= e + \frac{e^2}{4} \end{aligned}$$

Thus, we have shown that $k = e + \frac{e^2}{4}$.

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MathAA_11_SL_Summer_2021_Q6

Solution

1. Verification of the Trigonometric Identity

To show that $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$, we apply the **double-angle formulas** for sine and cosine.

- Recall the identities:

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = 1 - 2\sin^2 x$$

- Substitute these into the left-hand side (LHS) of the equation:

$$\begin{aligned}\text{LHS} &= \sin 2x + \cos 2x - 1 \\ &= (2\sin x\cos x) + (1 - 2\sin^2 x) - 1 \\ &= 2\sin x\cos x - 2\sin^2 x\end{aligned}$$

- Factor out the common term $2\sin x$:

$$\text{LHS} = 2\sin x(\cos x - \sin x)$$

  This matches the right-hand side (RHS), thus verifying the identity.

2. Solving the Trigonometric Equation

We are asked to solve $\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$ for $0 < x < 2\pi$.

- Using the result from part (a), we substitute $2\sin x(\cos x - \sin x)$ for the first three terms:

$$2\sin x(\cos x - \sin x) + (\cos x - \sin x) = 0$$

- Factor out the common binomial term $(\cos x - \sin x)$:

$$(\cos x - \sin x)(2\sin x + 1) = 0$$

- This gives us two cases to solve:

Case 1: $\cos x - \sin x = 0$

$$\sin x = \cos x$$
$$\tan x = 1$$

Within the interval $0 < x < 2\pi$, the **tangent function** is equal to 1 in the first and third quadrants:

$$x = \frac{\pi}{4}, \quad x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

Case 2: $2\sin x + 1 = 0$

$$2\sin x = -1$$
$$\sin x = -\frac{1}{2}$$

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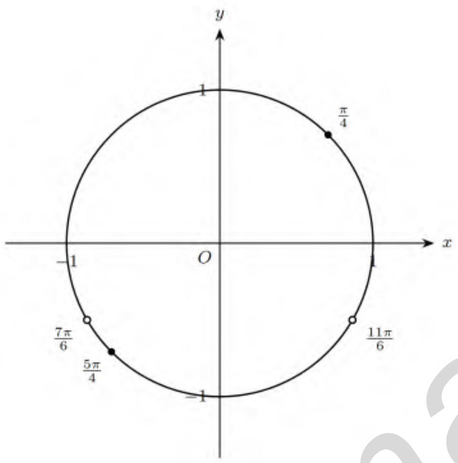
The **sine function** is equal to $-1/2$ in the third and fourth quadrants. The reference angle is $\pi/6$:

$$x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}, \quad x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$



- Collecting all solutions in the range $0 < x < 2\pi$:

$$x = \frac{\pi}{4}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{11\pi}{6}$$

$$\boxed{x = \frac{\pi}{4}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{11\pi}{6}}$$

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MathAA_11_SL_Summer_2021_Q7

Solution

1. Determination of the parameter m

To find the value of m, we consider the intersection of the **quadratic function** $f(x) = mx^2 - 2mx$ and the line $y = mx - 9$. The points of intersection are found by equating the two expressions:

$$mx^2 - 2mx = mx - 9$$

Rearranging this into the standard form of a **quadratic equation** $ax^2 + bx + c = 0$:

$$mx^2 - 3mx + 9 = 0$$

The problem states that the line meets the graph at exactly one point. This implies that the quadratic equation has exactly one real solution, which occurs when the **discriminant** (Δ) is equal to zero.

- For the equation $ax^2 + bx + c = 0$, the discriminant is $\Delta = b^2 - 4ac$.
- Here, $a = m$, $b = -3m$, and $c = 9$.

$$\begin{aligned}\Delta &= (-3m)^2 - 4(m)(9) \\ &= 9m^2 - 36m\end{aligned}$$

Setting the discriminant to zero:

$$\begin{gathered}9m^2 - 36m = 0 \\ 9m(m-4) = 0\end{gathered}$$

This gives two possible values for m: $m = 0$ or $m = 4$. If $m = 0$, $f(x) = 0$ (a horizontal line), and the line $y = -9$ would never meet it, which contradicts the "exactly one point" condition. Furthermore, for $f(x)$ to be a quadratic function, $m \neq 0$. Thus, we conclude:

$$\boxed{m = 4}$$

2. Factored form of the function

Given $m = 4$, the function is $f(x) = 4x^2 - 8x$. We want to express this in the form $f(x) = 4(x - p)(x - q)$.

- Factor out the common term $4x$:

$$\begin{aligned}f(x) &= 4x^2 - 8x \\ &= 4x(x-2)\end{aligned}$$

Comparing this to $4(x-p)(x-q)$, we identify the **roots** (or x-intercepts) as 0 and 2. Since the order of p and q is not specified:

$$\boxed{p = 0, q = 2 \ \ \text{(or vice versa)}}$$

3. Vertex form of the function

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We now express $f(x) = 4x^2 - 8x$ in the form $4(x-h)^2 + k$ by **completing the square**.

$$\begin{aligned} f(x) &= 4(x^2 - 2x) \\ &= 4(x^2 - 2x + 1 - 1) \\ &= 4((x-1)^2 - 1) \\ &= 4(x-1)^2 - 4 \end{aligned}$$

Comparing this to the form $4(x-h)^2 + k$, we find the coordinates of the **vertex** (h, k):

$$\boxed{h = 1, k = -4}$$

4. Analysis of function behavior

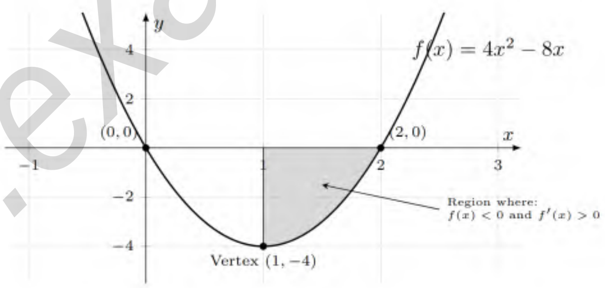
We seek the values of x where the graph of f is both negative ($f(x) < 0$) and increasing ($f'(x) > 0$).

- **Condition 1: $f(x) < 0$** From the factored form $f(x) = 4x(x-2)$, the parabola opens upwards (since $a = 4 > 0$) and has roots at $x = 0$ and $x = 2$. The function is negative between the roots:

$$0 < x < 2$$

- **Condition 2: $f(x)$ is increasing** A quadratic function $f(x) = a(x-h)^2 + k$ with $a > 0$ is increasing for $x > h$. From part (c), $h = 1$. Thus, the function is increasing when:

$$x > 1$$



- **Combined Interval:** We find the intersection of the two intervals $x \in (0, 2)$ and $x \in (1, \infty)$:

$$1 < x < 2$$

$$\boxed{1 < x < 2}$$

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MathAA_11_SL_Summer_2021_Q8

Solution

1. Differentiation of the function

To find the first derivative of $y = \frac{\ln x}{x^4}$, we apply the **Quotient Rule**, which states that for $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$. Let $u = \ln x$ and $v = x^4$. Then:

- $\frac{du}{dx} = \frac{1}{x}$
- $\frac{dv}{dx} = 4x^3$

Substituting these into the quotient rule formula:

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^4\left(\frac{1}{x}\right) - (\ln x)(4x^3)}{(x^4)^2} \\ &= \frac{x^3 - 4x^3\ln x}{x^8} \\ &= \frac{x^3(1-4\ln x)}{x^8} \\ &= \frac{1-4\ln x}{x^5}\end{aligned}$$

2. Finding the coordinates of point P

A **horizontal tangent** occurs where the first derivative is zero, $\frac{dy}{dx} = 0$.

- Setting the numerator of the derivative to zero:

$$\begin{aligned}1 - 4\ln x &= 0 \\ \ln x &= \frac{1}{4} \\ x &= e^{1/4}\end{aligned}$$

- To find the y-coordinate of P, substitute $x = e^{1/4}$ into $f(x)$:

$$\begin{aligned}y &= \frac{\ln(e^{1/4})}{(e^{1/4})^4} \\ &= \frac{1/4}{e} \\ &= \frac{1}{4e}\end{aligned}$$

The coordinates of P are $\boxed{(e^{1/4}, \frac{1}{4e})}$.

3. Classification of the stationary point

To show that P is a **local maximum**, we evaluate the second derivative $f''(x) = \frac{20\ln x - 9}{x^6}$ at $x = e^{1/4}$.

$$f''(e^{1/4}) = \frac{20\ln(e^{1/4}) - 9}{(e^{1/4})^6}$$
$$= \frac{20(\frac{1}{4}) - 9}{e^{3/2}}$$
$$= \frac{5-9}{e^{3/2}}$$
$$= -\frac{4}{e^{3/2}}$$

Since $f''(e^{1/4}) < 0$, the **Second Derivative Test** confirms that point P is a local maximum.

4. Solving the inequality

We solve $f(x) > 0$ for $x > 0$:

$$\frac{\ln x}{x^4} > 0$$

Since $x^4 > 0$ for all $x > 0$, the inequality depends only on the numerator:

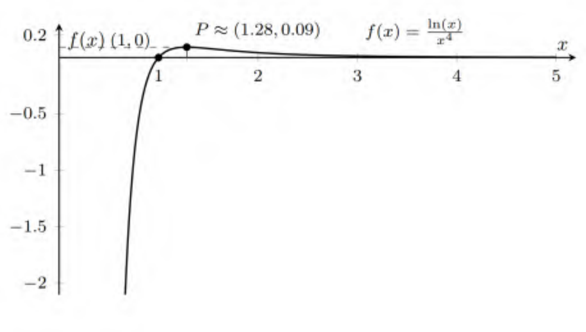
$$\ln x > 0 \implies x > e^0 \implies x > 1$$

The solution is $\boxed{x > 1}$.

5. Graph of the function

The graph of $f(x) = \frac{\ln x}{x^4}$ features:

- An x-intercept at $(1, 0)$.
- A local maximum at $P(e^{1/4}, \frac{1}{4e}) \approx (1.28, 0.092)$.
- A **vertical asymptote** at $x = 0$ (as $x \to 0^+$, $f(x) \to -\infty$).
- A **horizontal asymptote** at $y = 0$ (as $x \to \infty$, $f(x) \to 0$ by **L'Hôpital's Rule**).



MathAA_11_SL_Summer_2021_Q9

Solution

Based on the provided probability distributions for the two biased four-sided dice, A and B, we can determine the requested values and ranges by applying the fundamental properties of **probability distributions** and **expected value**.

1. Analysis of Die A The sum of probabilities for any discrete random variable must equal 1. For die A, the score X has the following distribution:

- $P(X = 1) = p$
- $P(X = 2) = p$
- $P(X = 3) = p$
- $P(X = 4) = \frac{1}{2}p$

(a) To find p:

$$\begin{aligned} p + p + p + \frac{1}{2}p &= 1 \\ 3.5p &= 1 \\ p &= \frac{1}{3.5} = \frac{2}{7} \end{aligned}$$

$$\boxed{p = \frac{2}{7}}$$

(b) To find $E(X)$:

$$\begin{aligned} E(X) &= \sum x \cdot P(X = x) \\ &= 1(p) + 2(p) + 3(p) + 4\left(\frac{1}{2}p\right) \\ &= p + 2p + 3p + 2p = 8p \\ &= 8\left(\frac{2}{7}\right) = \frac{16}{7} \end{aligned}$$

$$\boxed{E(X) = \frac{16}{7}}$$

2. Analysis of Die B For die B, the score Y has the distribution:

- $P(Y = 1) = q$
- $P(Y = 2) = q$
- $P(Y = 3) = q$
- $P(Y = 4) = r$

The sum of probabilities is $3q + r = 1$, which implies $r = 1 - 3q$.

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(c) (i) Since r is a probability, $0 \leq r \leq 1$. However, for q to also be a valid probability ($q \geq 0$), the maximum value of r occurs when $q = 0$ ($r = 1$). The minimum value of r occurs when q is at its maximum. Since $3q \leq 1$, $q \leq \frac{1}{3}$, which leads to $r \geq 0$. $\boxed{0 \leq r \leq 1}$

(ii) From $3q + r = 1$, we have $q = \frac{1-r}{3}$.

- If $r = 0$, $q = \frac{1}{3}$.
- If $r = 1$, $q = 0$. $\boxed{0 \leq q \leq \frac{1}{3}}$

(d) The expected value $E(Y)$ is:

$$\begin{aligned} E(Y) &= 1(q) + 2(q) + 3(q) + 4(r) \\ &= 6q + 4(1 - 3q) \\ &= 6q + 4 - 12q = 4 - 6q \end{aligned}$$

Using the range $0 \leq q \leq \frac{1}{3}$:

- If $q = 0$, $E(Y) = 4$.
- If $q = \frac{1}{3}$, $E(Y) = 4 - 6\left(\frac{1}{3}\right) = 4 - 2 = 2$. $\boxed{2 \leq E(Y) \leq 4}$

3. Game Probability and Final Calculation Agnes rolls die A and Barbara rolls die B. We are given $P(X < Y) = \frac{1}{2}$. The possible outcomes where $X < Y$ are:

- $X = 1$ and $Y \in \{2, 3, 4\}$
- $X = 2$ and $Y \in \{3, 4\}$
- $X = 3$ and $Y \in \{4\}$

(e) Calculating $P(X < Y)$:

$$\begin{aligned} P(X < Y) &= P(X = 1)P(Y > 1) + P(X = 2)P(Y > 2) + P(X = 3)P(Y > 3) \\ \frac{1}{2} &= p(q + q + r) + p(q + r) + p(r) \\ \frac{1}{2} &= p(2q + r) + p(q + r) + pr \\ \frac{1}{2} &= p(3q + 3r) \end{aligned}$$

Substitute $p = \frac{2}{7}$ and $r = 1 - 3q$:

$$\begin{aligned} \frac{1}{2} &= \frac{2}{7}(3q + 3(1 - 3q)) \\ \frac{1}{2} &= \frac{2}{7}(3q + 3 - 9q) \\ \frac{1}{2} &= \frac{2}{7}(3 - 6q) \\ \frac{7}{4} &= 3 - 6q \\ 6q &= 3 - \frac{7}{4} = \frac{5}{4} \\ q &= \frac{5}{24} \end{aligned}$$

Now, substitute $q = \frac{5}{24}$ into the expression for $E(Y)$:

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$$\begin{aligned} E(Y) &= 4 - 6q \\ &= 4 - 6\left(\frac{5}{24}\right) \\ &= 4 - \frac{5}{4} = \frac{11}{4} \end{aligned}$$

$$\boxed{E(Y) = 2.75}$$

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